Structural instability of Friedmann-Robertson-Walker cosmological models

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Abstract

Cosmological singularity and asymptotic behaviour of scale factor of generalized cosmological models are analyzed in respect of their structural stability. It is shown, that cosmological singularity is structurally unstable for the majority of models with barotropic perfect fluid with strong energy condition. Inclusion of Λ-term extends the set of structurally stable cosmological models.

1   Introduction

Relativistic Friedmann-Robertson-Walker cosmological models (FRW-models) perform generally accepted current theoretical basis for description of global structure and evolution of the Universe. Remarkable property of these models is their simplicity: in most situations we deal with one scale factor and equation of state, which has to be specified by some additional physical considerations. Observational data allow in principle to correct some basic parameters of the models, as well as to specify properties of matter at different stages of evolution of the Universe and sometimes they imply reconsideration of Einstein gravity in the context of some generalized theories [1, 2, 3]. History of cosmology shows (and methodological principle of eligibility prescribes) that correction of standard cosmological models is realized mainly by sequence of their, in some sense, "small" modifications. In this way we are necessarily faced with situation which is well known in qualitative differential equations theory [4]: some properties of original (non-modified) model may "survive" after "small" modification of the model, while others may disappear. In the former case the property is referred to as "rough" or structurally stable, in the latter case — as "thin" or structurally unstable. As an example of such thin property of FRW-models we may consider cosmological singularity. By Hawking-Penrose theorem [5] which has been proved within Einstein GR, collapse is inevitably reached at some finite moment of time. However, there are many exact cosmological solutions, obtained in frame of generalized theories of gravitation (for example, involving scalar fields or non-riemannian objects [6, 7]), which describe evolution of the Universe without singularity or with singularities of qualitatively different kinds. By general considerations consecutive cosmological model should have structurally stable (with respect to possible modifications) basic properties. In the opposite case any "small" work modification of the theory will almost always lead to a change of cosmological paradigm.

In present paper we perform analysis of structural stability of FRW-models. We restrict ourselves by investigations of structural perturbations induced by generalized $f(R)$-theories of gravity and nonlinear equations of state. Both of these generalizations of standard cosmology have been actively investigated last decade in the context of new cosmological observational data [8, 9, 10, 11, 12, 13, 18]. We investigate structural stability of the two important properties of FRW-models: cosmological singularity (S-stability) and asymptotic behaviour of scale factor at $t \to \infty$ (A-stability). Latter property makes sense only for open and flat FRW-models. These restrictions allow to reduce the problem to investigation of asymptotic

solutions to second order differential equation with variable coefficients. In spite of quite different motivation of this work, our analysis in some aspects reproduces and supplements results obtained in [11, 14, 15, 16, 17]. In some important points we go to the conclusions made earlier in [16, 17].

In section 2 we outline class of standard models. In section 3 we describe those generalized models which will be used for investigation of structural stability of standard FRW-models. Basic equation on structural perturbations and its analysis are performed in section 4. Section 5 is devoted to investigation of structural stability of standard FRW-models with Λ-term. Here we apparently illustrate the idea of "structural security" of cosmological model. In Conclusion we summarize and discuss all obtained results.

2   Standard FRW-models

Standard homogeneous isotropic cosmological models can be described by metric of the following kind:

$$g = N^2(t)dt \otimes dt - a^2(t)(dr \otimes dr + \frac{\sin^2(kr)}{k^2}(d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi)), \tag{1}$$

where $N(t)$ is factor, defining gauge of cosmological time, $a(t)$ is scale factor, describing cosmological evolution of space lengths, k is curvature constant parameter, r is radial coordinate, θ and φ are angle coordinates. The parameter k has dimension of inverse length and can be real, imaginary or null. In the first case we deal with closed cosmological models, in the second case — with open cosmological models, in the third case — with flat ones. In what follows all formulas will have universal sense for all possible values of parameter k. The case $k = 0$ sometimes will require rather simple limit procedure: $k \to 0$.

Standard calculations give the following nonzero components of mixed Ricci tensor:

$$R_0^0 = \frac{3}{N^2}\left(\frac{\dot{N}\dot{a}}{Na} - \frac{\ddot{a}}{a}\right); \quad R_1^1 = R_2^2 = R_3^3 = \frac{1}{N^2}\left(\frac{\dot{N}\dot{a}}{Na} - \frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2}\right) - 2\frac{k^2}{a^2}. \tag{2}$$

Scalar curvature is expressed by the formula[1]:

$$R \equiv R_\alpha^\alpha = \frac{6}{N^2}\left(\frac{\dot{N}\dot{a}}{Na} - \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - 6\frac{k^2}{a^2}. \tag{3}$$

It is commonly accepted, that source of gravity in cosmology is barotropic perfect fluid with energy-momentum tensor of the kind:

$$T = (\varepsilon + p)u \otimes u - pg, \tag{4}$$

where $u = Ndt$ — 1-form of 4-velocity of matter, defining comoving reference frame, ε and p — scalars of energy density and pressure of the matter, connected with each other by barotropic equation of state:

$$p = \alpha\varepsilon, \quad \text{where} \quad \alpha = \text{const.} \tag{5}$$

Apart from perfect fluid for the more generality of our formulas we include into the standard models $\Lambda-$term (which, excepting section 5, we shall assume zero). Einstein equations:

$$G_\beta^\alpha \equiv R_\beta^\alpha - \frac{1}{2}\delta_\beta^\alpha R = \kappa T_\beta^\alpha + \Lambda\delta_\beta^\alpha, \tag{6}$$

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[1]Hereafter Greek indexes α, β, γ denote space-time components of 4-tensors, Latin indexes i, j, k — their space components.

derived from Einstein-Hilbert variational principle for the action (S_{m} — action for matter, $\kappa = 8\pi G/c^4$ — Einstein gravitational constant, $*$ — standard dual conjugation):

$$S[g] = -\frac{1}{2\kappa}\int *(R + 2\Lambda) + S_{\mathrm{m}}, \tag{7}$$

take the form of basic dynamical equations:

$$\frac{3\dot{a}^2}{N^2 a^2} + \frac{3k^2}{a^2} = \kappa\epsilon + \Lambda; \quad \frac{1}{N^2}\left(-\frac{2\dot{N}\dot{a}}{Na} + 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{k^2}{a^2} = -\kappa p + \Lambda. \tag{8}$$

For the simplicity sake we accept conformal gauge of cosmological metric (1), which is defined by condition: $N = a$. Also we go to units, where $\kappa = 1$ and $c = 1$. For chosen gauge formulas (2)-(3) take the form:

$$R_0^0 = \frac{3}{a^2}\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right); \quad R_1^1 = R_2^2 = R_3^3 = -\frac{1}{a^2}\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} + 2k^2\right); \quad R = -\frac{6}{a^3}\left(\ddot{a} + k^2 a\right), \tag{9}$$

and Einstein equations (8) are expressed by the formulas:

$$\frac{3}{a^2}\left(\frac{\dot{a}^2}{a^2} + k^2\right) = \varepsilon + \Lambda; \quad \frac{1}{a^2}\left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + k^2\right) = -p + \Lambda. \tag{10}$$

With using (10) it is easily to check validity of the relation:

$$\dot{\varepsilon} = -3(p + \varepsilon)\frac{\dot{a}}{a}, \tag{11}$$

having sense of energy conservation in adiabatically evolving Universe. For equation of state (5) relation (11) leads to the integral:

$$\varepsilon = ca^{-3(1+\alpha)}, \tag{12}$$

where c — integration constant. Combining (12) and first equation in (10), we go to the useful formula:

$$\dot{a} = a\sqrt{\frac{ca^{-1-3\alpha}}{3} + \frac{\Lambda a^2}{3} - k^2}. \tag{13}$$

Differentiating this formula and expressing derivative $\dot{a}$ again through this formula, we go to another useful formula:

$$\ddot{a} = \frac{c(1 - 3\alpha)}{6}a^{-3\alpha} - k^2 a + \frac{2}{3}\Lambda a^3. \tag{14}$$

By the same manner one can obtain the following useful formulas, which will be used in next sections:

$$R = (3\alpha - 1)ca^{-3(1+\alpha)} - 4\Lambda; \quad R_0^0 = \frac{c(1 + 3\alpha)}{2}a^{-3(1+\alpha)} - \Lambda; \quad R_i^i = \frac{c(\alpha - 1)}{2}a^{-3(1+\alpha)} - \Lambda; \tag{15}$$

$$\dot{R} = -3c(1 + \alpha)(3\alpha - 1)a^{-3(1+\alpha)}\sqrt{\frac{ca^{-1-3\alpha}}{3} + \frac{\Lambda a^2}{3} - k^2}; \tag{16}$$

$$\ddot{R} = -3c(1 + \alpha)(3\alpha - 1)a^{-3(1+\alpha)}(3(1 + \alpha)k^2 - \frac{c(7 + 9\alpha)}{6}a^{-1-3\alpha} - \frac{1}{3}(3\alpha + 2)\Lambda). \tag{17}$$

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Equation (13) under $\Lambda = 0$ can be integrated in elementary functions[2]. The result has the form:

$$a(t) = \begin{cases} (\sqrt{c/3}\sin[k(1+3\alpha)t/2]/k)^{2/(1+3\alpha)}, & \alpha \neq -1/3; \\ \\ a_0\exp[\pm\sqrt{c/3-k^2}t], & \alpha = -1/3. \end{cases} \tag{18}$$

Simple analysis of the integral (13) shows, that in physical region ($\alpha \geq -1$) $\Lambda-$term plays no role in vicinity of cosmological singularity, but it dominates under $a \to \infty$ for flat and open cosmological models.

At the end of this section we perform nonzero components of second covariant derivatives (acting on scalar functions) $\nabla_{\alpha\beta} \equiv \nabla_\alpha\nabla_\beta$ in FRW-models:

$$\nabla_{00} = \frac{d^2}{dt^2} - \frac{\dot{a}}{a}\frac{d}{dt}; \quad \nabla_{ik} = g_{ik}\frac{\dot{a}}{a^3}\frac{d}{dt}; \quad \Box \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\sqrt{-g}g^{\alpha\beta}\frac{\partial}{\partial x^\beta} = \frac{1}{a^2}\left(\frac{d^2}{dt^2} + \frac{2\dot{a}}{a}\frac{d}{dt}\right). \tag{19}$$

3 Generalized cosmological models

In present paper we consider generalized cosmological models of the following two kinds:

1. Models with nonlinear equation of state;

2. Models with nonlinear on curvature action for gravity.

3.1 Nonlinear equation of state

Lets consider barotropic fluid with equation of state of the following kind:

$$p = \alpha\varepsilon(1 + \sigma\varepsilon^b), \tag{20}$$

where σ and b are parameters, responsible for nonlinear properties of the matter. Under $\sigma \to 0$ equation (20) goes[3] to the standard equation (5). Physical nature of the nonlinearity can be concerned with the more complicated mechanisms of interaction of matter particles, quantum effects, unknown fundamental interactions, etc.

If energy density of such nonlinear matter has structural perturbation $\delta\varepsilon$, then its pressure will have structural perturbation of the kind:

$$\delta p = \alpha\delta\varepsilon + \alpha\sigma\varepsilon^{b+1}. \tag{21}$$

In this expression nonlinear term is not varied, since we assume, that it already has first order of smallness (with respect to σ).

3.2 Nonlinear models of gravity

Lets consider general scheme of nonlinear theories of gravity in cosmological context. For space-time of FRW-models there is the only scalar invariant — scalar curvature R. So, general kind of action for nonlinear cosmological models, generalizing Einstein-Hilbert action, can be expressed by the formula:

$$S_g = -\frac{1}{2}\int *f(R), \tag{22}$$

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[2]Though integral for world time: $\tau = \int_0^t a(t)\,dt$ can be expressed in elementary functions only for some particular values of parameters α and k.

[3]For the case of dust-like matter $\alpha = 0$ it is necessary to use the following limit procedure in (20): $\alpha \to 0$, $\sigma \to \infty$, $\alpha\sigma < \infty$.

where $f(R)$ — arbitrary function of scalar curvature. Standard variational procedure[4] leads to the following expression for variational derivative of (22) with respect to metric:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{g}}}{\delta g^{\alpha\beta}} \equiv -\frac{1}{2}\mathcal{G}_{\alpha\beta} = -\frac{1}{2}([R_{\alpha\beta} - \nabla_\alpha\nabla_\beta + g_{\alpha\beta}\square]f'(R) - \frac{1}{2}g_{\alpha\beta}f(R)). \tag{23}$$

where $\mathcal{G}$ is generalized Einstein tensor. Let $f(R)$ can be expressed through the generalized Taylor decomposition:

$$f(R) = 2\Lambda + R + \sum_{s\neq 0,1}\lambda_s R^s, \tag{24}$$

where $2\Lambda = \lambda_0$, $\lambda_1 = 1$ and s — any real number, excepting 0 and 1. Choice of the λ_1 corresponds to our main problem: to investigate structural stability of cosmological models in vicinity of standard FRW-models. Choice of the λ_0 will allow us to investigate structural stability of Λ-term cosmology in frame of general scheme. So, in space of nonlinear theories of gravity we are interested by those theories, which are characterized by the following infinitely dimensional vector of parameters: $(\ldots\lambda_{s_{-1}}, \lambda_0, 1, \lambda_{s_2}, \ldots)$, when [5] all $\lambda_{s_i} \to 0$ under $i \neq 0, 1$. In view of this aim it is expediently to decompose (with using (24)) tensor $\mathcal{G}$ on standard Einstein tensor (6) and "perturbing" part, which can be interpreted (up to a constant) as energy-momentum tensor T^{n} of nonlinear self-action:

$$\mathcal{G}_{\alpha\beta} = G_{\alpha\beta} + \Delta G_{\alpha\beta} = G_{\alpha\beta} + [\mathring{R}_{\alpha\beta} - \mathring{\nabla}_\alpha\mathring{\nabla}_\beta + \mathring{g}_{\alpha\beta}\mathring{\square}]F'(\mathring{R}) - \frac{1}{2}\mathring{g}_{\alpha\beta}F(\mathring{R}), \tag{25}$$

where

$$F(\mathring{R}) = 2\Lambda + \sum_{s\neq 0,1}\lambda_s \mathring{R}^s,$$

and circle over values relates them to unperturbed cosmological metric of those FRW-model, whose structural stability we are studying.

Typical s-th member of this nonlinear perturbation in right-hand side of perturbed Einstein equation will have the following kind:

$$-(\Delta G_s)^\alpha_\beta = (T^{\mathrm{n}}_s)^\alpha_\beta = -s\lambda_s(\mathring{R}^\alpha_\beta - \mathring{\nabla}^\alpha\mathring{\nabla}_\beta + \delta^\alpha_\beta\,\mathring{\square})\,\mathring{R}^{s-1} + \frac{\lambda_s}{2}\delta^\alpha_\beta\mathring{R}^s. \tag{26}$$

With using (9), (12)-(17) and (19) after some algebra, we go to the following expressions for non-zero components of nonlinear self-action tensor (there is no summation over repeating indexes!):

$$(T^{\mathrm{n}}_s)^\beta_\beta = \lambda_s(3\alpha - 1)^{s-1}c^s a^{-3s(1+\alpha)}(A^\beta_s + B^\beta_s a^{1+3\alpha}), \tag{27}$$

where

$$A^0_s = \frac{1}{2}(3\alpha(2s^2 - 3s + 1) + 6s^2 - 7s - 1); \quad B^0_s = -\frac{9k^2}{c}s(s-1)(1+\alpha); \tag{28}$$

$$A^1_s = A^2_s = A^3_s = -(s + s\alpha - 1)A^0_s; \quad B^1_s = B^2_s = B^3_s = -(s + s\alpha - \alpha - \frac{4}{3})B^0_s \tag{29}$$

Note, that all formulas are valid for any real value of parameter s.

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[4]Under varying Lagrange density $f(R)\sqrt{-g}$ it is helpful to use Palatini's identity: $\delta R_{\alpha\beta} = \nabla_\gamma\delta\Gamma^\gamma_{\alpha\beta} - \nabla_\beta\delta\Gamma^\gamma_{\alpha\gamma}$.

[5]Note, that all λ_s have different dimensions: $[\lambda_s] = \ell^{2(s-1)}$. In spite of conditional sense of "smallness" for any dimensional value, our analysis in vicinity of the point $(\ldots, 0, 1, 0, \ldots)$ in parametric space is absolutely correct.

4 Equations for structural perturbations

In order to derive linear differential equation for structural perturbations it is necessary to extract null order in the left-hand (geometrical) side of structurally perturbed Einstein equations:

$$\hat{G}(a(t)) = T^{\mathrm{n}} + T^{\mathrm{mat}}, \tag{30}$$

where we consider now Einstein tensor as nonlinear differential operator $\hat{G}$, acting on space of scale factors. Introducing structural perturbation of scale factor: $a(t) \to a(t) + \delta(t)$, we go to the formula:

$$\hat{G}(a(t) + \delta(t)) = \hat{G}(a(t)) + \delta\hat{G}(\delta(t), a(t)) + o(\delta), \tag{31}$$

where

$$\delta\hat{G}(\delta(t), a(t)) \equiv \frac{d}{d\varepsilon}\hat{G}(a(t) + \epsilon\delta(t))|_{\epsilon=0} \tag{32}$$

— "differential of Einstein operator", which is linear differential operator on its first argument. Direct calculations with using formulas (10), (32) and (12)-(14) lead to the following nonzero components of δG:

$$\delta G^0_0(\delta, a) = \frac{6}{a^3}\left(\sqrt{ca^{-1-3\alpha}/3 - k^2}\dot{\delta} + (k^2 - 2ca^{-1-3\alpha}/3)\delta\right); \tag{33}$$

$$\delta G^1_1(\delta, a) = \frac{2}{a^3}\ddot{\delta} - \frac{2}{a^3}\sqrt{ca^{-1-3\alpha}/3 - k^2}\dot{\delta} + \frac{c(1+9\alpha)}{3}a^{-4-3\alpha}\delta. \tag{34}$$

Substituting it into equations (30) and keeping in mind, that in null order on δ these equations lead to Einstein equations (10) for unperturbed FRW-model which are satisfied identically, we go to the following linearized system of equations on structural perturbations:

$$\frac{6}{a^3}\left(\sqrt{ca^{-1-3\alpha}/3 - k^2}\dot{\delta} + (k^2 - 2ca^{-1-3\alpha}/3)\delta\right) = (T^{\mathrm{n}})^0_0 + \delta\varepsilon; \tag{35}$$

$$\frac{2}{a^3}\ddot{\delta} - \frac{2}{a^3}\sqrt{ca^{-1-3\alpha}/3 - k^2}\dot{\delta} + \frac{c(1+9\alpha)}{3}a^{-4-3\alpha}\delta = (T^{\mathrm{n}})^1_1 - \delta p. \tag{36}$$

Using formula (21) and excluding from (35)-(36) structural perturbation $\delta\varepsilon$, we obtain unique basic equation:

$$2\ddot{\delta} + 2(3\alpha - 1)\sqrt{ca^{-1-3\alpha}/3 - k^2}\dot{\delta} + (6\alpha k^2 + c(1-3\alpha)a^{-1-3\alpha}/3)\delta = \tag{37}$$

$$a^3(\alpha(T^{\mathrm{n}})^0_0 + (T^{\mathrm{n}})^1_1 - \alpha\sigma\varepsilon^{b+1}).$$

Going to the new variable $x = a(t)$, equation (37) can be transformed to the form:

$$2x^2(cx^{-1-3\alpha}/3 - k^2)\delta'' + (\delta - x\delta')(6\alpha k^2 + c(1-3\alpha)x^{-1-3\alpha}/3) = \tag{38}$$

$$x^3(\alpha(T^{\mathrm{n}})^0_0 + (T^{\mathrm{n}})^1_1) - \alpha\sigma c^{b+1}x^{-3(\alpha+1)(b+1)+3}.$$

Since sources of structural perturbations in first order act independently from each other, we can analyze structural stability of FRW-models by means of equation (38) separately for every source. Limit correspondence considerations suggest that structural perturbations of scale factor must be determined by those particular solutions of differential equation (38) which vanish when source of perturbations vanishes[6].

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[6]So, we omit general solutions to homogeneous equation (38) (without its right-hand side). It is interesting to note, that this solutions define class of infinitesimal *fantom* conformal perturbations, which up to high orders of smallness transform apparently different metrics, originated from the same matter source, to each other.

4.1 Structural stability of cosmological models within nonlinear gravity

Let consider structural perturbations of $f(R)$-cosmological models. Perturbation from s-th member of Taylor decomposition in our notations takes the form:

$$x^3(\alpha((T_s^n)_0^0 + (T_s^n)_1^1) = \lambda_s(3\alpha - 1)^{s-1}c^s x^{-3s(1+\alpha)+3}(A_s + B_s x^{1+3\alpha}), \tag{39}$$

where

$$A_s = \alpha A_s^0 + A_s^1 = (\alpha+1)(1-s)A_s^0; \quad B_s = \alpha B_s^0 + B_s^1 = (2\alpha + \frac{4}{3} - s - s\alpha)B_s^0.$$

Looking at equation (38) with right-hand side (39) together with (28)-(29), one can immediately conclude: *all standard cosmological models with isotropic radiation ($\alpha = 1/3$) and vacuum-like matter ($\alpha = -1$) are structurally stable in any order on curvature,* since this concrete values of α imply vanishing of sources of structural perturbations.

In general case let introduce the following notations:

$$\xi(x) = \frac{\delta(x)}{3\lambda_s(3\alpha - 1)^{s-1}c^{s-1}(1+\alpha)(1-s)x}; \quad \bar{k}^2 = \frac{3k^2}{c}.$$

General condition of structural stability is *boundedness of relative perturbation $\xi(x)$ for any allowed x.* Note, that in this section we accept, that $\alpha \neq -1$ and $\alpha \neq 1/3$ in view of above mentioned specific properties of FRW-models with such parameters. Also we put anywhere[7] $s \neq 1$.

In order to analyze S-stability let consider asymptotic kind of equation (38) together with its right-hand side (39) under $x \to 0$ and under $\alpha > -1/3$, when cosmological singularity does take place (solution (18)). For general values of parameters this asymptotic kind is expressed by the following equation:

$$2x^2(x\xi(x))'' + (1-3\alpha)x(\xi(x) - (x\xi(x))') = \frac{1}{2}(3\alpha(2s^2 - 3s + 1) + 6s^2 - 7s - 1)x^{-3s(1+\alpha)+4+3\alpha}. \tag{40}$$

Its particular solution, which we are interested, has the form:

$$\xi(x)|_{x\to 0} = \frac{6\alpha s^2 - 9s\alpha + 3\alpha + 6s^2 - 7s - 1}{6(6s + 6s\alpha - 7 - 9\alpha)(1+\alpha)(s-1)}x^{-3(s-1)(1+\alpha)}. \tag{41}$$

So, condition of S-stability is positiveness of index: $-3(s-1)(1+\alpha) > 0$, that under $1+3\alpha > 0$ takes place, when $s < 1$. In other words, *almost all nonlinear analytic generalizations of standard cosmology possess structural instability of singularity.* Note, that non-analytic nonlinear models (for example, containing in lagrangians negative powers of R) may be S-stable.

The words "almost all" are necessary to separate Λ-term cosmology ($s = 0$) and some particular cases, which are S-stable. S-stability of the particular case, when coefficient in (41) vanishes:

$$6\alpha s^2 - 9s\alpha + 3\alpha + 6s^2 - 7s - 1 = 0, \tag{42}$$

corresponds to the asymptotic equation (40) with other right-hand side:

$$2x^2(x\xi(x))'' + (1-3\alpha)x(\xi(x) - (x\xi(x))') = 3\bar{k}^2 s(2\alpha + \frac{4}{3} - s(1+\alpha))x^{-3s(1+\alpha)+5+6\alpha}. \tag{43}$$

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[7]Situation with $s = 1$ has no particular interest, since, in fact, it can be reduced to redefinition of Einstein gravitational constant and so it can not influence on stability.

Using relation between parameters:

$$\alpha = -\frac{6s^2 - 7s - 1}{3(2s^2 - 3s + 1)}, \tag{44}$$

which follows from (42), we obtain the following particular solution to equation (43):

$$\xi(x)|_{x \to 0} = -\frac{\bar{k}^2 s(2s-1)(s-1)}{2(4s^2 - 7)} x^{-2(s^2 + s - 3)/(2s-1)(s-1)}. \tag{45}$$

Omitting the situations with $s < 1$, when almost all models are S-stable, we obtain two new classes of S-stable models. The first one is obtained under $s \in (1; s_*]$, that corresponds to monotonic variation of $\alpha \in (\infty; \alpha_*]$. Values of boundaries of the intervals are:

$$s_* = \frac{-1 + \sqrt{13}}{2} \approx 1.30; \quad \alpha_* = -\frac{47 - 13\sqrt{13}}{3(19 - 5\sqrt{13})} \approx -0.04. \tag{46}$$

Second class is obtained for flat models $\bar{k} = 0$. The interval of nontrivial S-stability (i.e., interval, where $\alpha > -1/3$ is described by the formula (44) for $s \in (1; (1 + \sqrt{3})/2)$.

Finally, the last particular case corresponds to the following relation between parameters:

$$(6s + 6s\alpha - 7 - 9\alpha) = 0 \Leftrightarrow \alpha = -\frac{6s - 7}{3(2s - 3)} \Leftrightarrow s = \frac{7 + 9\alpha}{6(1 + \alpha)}, \tag{47}$$

when denominator in (41) vanishes. Asymptotic particular solution in this case has the form:

$$\xi(x)|_{x \to 0} = \frac{(1 - 3\alpha)((1 + 3\alpha)\ln(x) + 2)}{2(1 + 3\alpha)^2} x^{-(1+3\alpha)/2}. \tag{48}$$

Under $\alpha > -1/3$ $(1 < s < 3/2)$ power is negative and the models are S-unstable, while under $s < 1$ there is no singularity and the models are S-stable.

So far the case $\alpha = -1/3$ has remained out of our consideration. Particular solution to the equation (38) with right-hand side (39), taken under $\alpha = -1/3$, has the form:

$$\xi(x) = \frac{-2s^2 + 2s + 1 - 2\bar{k}^2 s + 2\bar{k}^2 s^2}{8(\bar{k}^2 - 1)(s - 1)^2} x^{-2s+2}, \tag{49}$$

that leads to the conclusion: *under $s > 1$ generalized nonlinear models with equation of state: $p = -\varepsilon/3$ are S-unstable and A-stable* and vise verse: *under $s < 1$ such models are S-stable and A-unstable.* Particular situation arises when coefficient in expression (49) vanishes. In this case the model is structurally stable in all senses, at the least, in linear approximation. In closed models this situation is described by relation:

$$s = \frac{\bar{k}^2 - 1 \pm \sqrt{\bar{k}^4 - 4\bar{k}^2 + 3}}{2(\bar{k}^2 - 1)}. \tag{50}$$

Since solution (18) must be real, parameter $\bar{k}$ lies in interval $(0; 1)$. Parameter s (on both branches of square root) belongs to the union $(-\infty; (1 - \sqrt{3})/2] \cup [(1 + \sqrt{3})/2; \infty)$. For open models relation between s and $\bar{k}$ can be obtained from (50) by the substitution $\bar{k}^2 \to -\bar{k}^2$. When $\bar{k}$ varies from 0 to ∞ parameter s monotonically falls from $(1 + \sqrt{3})/2$ to 1 on upper branch of square root and monotonically increases from $(1 - \sqrt{3})/2$ to 0 on lower one. The value $\bar{k}^2 = 1$ is singular for equation (38), since under $\alpha = -1/3$ and $\bar{k}^2 = 1$ this equation become algebraic: $x^{-2s+3} = 0$. It corresponds to the limit $s \to \infty$, which is reached on the upper branch of solution (50). In this degenerate case cosmological models can be both structurally stable, and structurally unstable, since basic equation (38) is satisfied by any function $\delta(x)$. All results are summarized in the diagram of S-stability (fig.1).
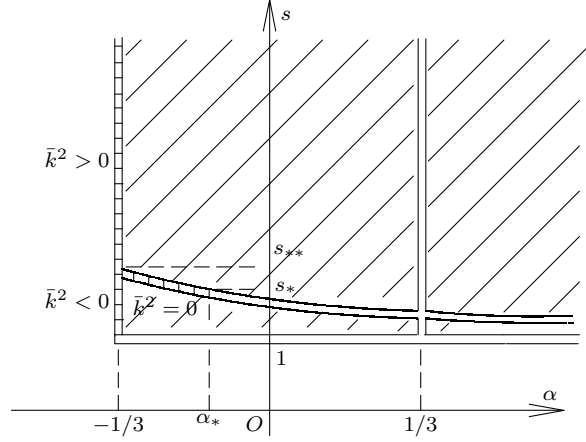
Figure 1: Diagram of S-stability of FRW-models. Inclined dash points region of S-instability of FRW-models. Boundary of this region is double line. At this boundary FRW-models can be both S-stable (vertical line $\alpha = 1/3$, horizontal line $s = 1$ and part of rational dependency (44), beginning at the point $\{s_*; \alpha_*\}$ (formula (46)) and conditionally S-stable (vertical line (with horizontal dash) $\alpha = -1/3$ with condition (50); upper part of this line correspondes to closed models, lower — to the open ones; and part (with vertical dash) of dependency (44), lying between s_* and $s_{**} = (1 + \sqrt{3})/2$ with condition $\bar{k}^2 = 0$).

In order to analyze A-stability of FRW-models it is necessary to go to asymptotic kind of equation (38) and of its right-hand side (39) under $x \to \infty$. Note, that energetic condition: $1 + 3\alpha \geq 0$ is unnecessary now.

In first, lets note, that under $1 + 3\alpha < 0$ asymptotic kind of equation for structural perturbations exactly coincides with (40), while its particular solution coincides with (41), which one should consider now as solution under $x \to \infty$. Analysis of power sign leads to the conclusion, that *under $-1 < \alpha < -1/3$ almost all FRW-models are A-stable when $s > 1$ and are A-instable when $s < 1$, and under $\alpha < -1$ vise verse: almost all FRW-models are A-instable when $s > 1$ and A-stable when $s < 1$.* Lets go to the particular situations. The case, when coefficient in (41) vanishes is described by the formulas (42)-(45), which should be considered as asymptotic expressions under $x \to \infty$ and $\alpha < -1/3$. Additional A-stable branch of FRW-models arises here when $s \in (-\infty; s^*] \cup (2; \infty)$, where $s^* = -(1 + \sqrt{13})/2 \approx -2.30$. Here on the first interval $\alpha \in (-1; \alpha^*]$, where $\alpha^* = -(4 + \sqrt{13})/9 \approx -0.85$ and α monotonically increases. On the second interval α decreases from -1 to its minimal value $\alpha_{\min} = (4\sqrt{6} - 13)/3 \approx -1.07$, which is reached under $s_{\min} = 2 + \sqrt{6}/2 \approx 3.22$, and then monotonically increases to the value -1. There is no new A-stable models in case of vanishing of denominator in (41).

Now lets go to the case $1 + 3\alpha > 0$. Asymptotic kind of equation for structural perturbations under $\bar{k}^2 \neq 0$ takes the form:

$$-2x^2(x\xi(x))'' + 6\alpha x(\xi(x) - (x\xi(x))') = 3s(2\alpha + \frac{4}{3} - s(1 + \alpha))x^{-3s(1+\alpha)+4+3\alpha}. \quad (51)$$

Its particular solution:

$$\xi(x)|_{x\to\infty} = \frac{sx^{-3(s-1)(1+\alpha)}}{6(s-1)(1+\alpha)}. \quad (52)$$

So, *under $\alpha > -1/3$ almost all non-flat FRW-models are A-stable under $s > 1$ and are A-unstable under $s < 1$.* In particular case $s = 0$ equation (51) will have another right-hand side and another particular solution:

$$\xi(x)|_{x\to\infty} = -\frac{3\alpha - 1}{24q^2(1 + \alpha)}x^2,$$

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that implies A-instability of all such models.

Finally, for the case of flat models equation for structural perturbation become identical to (40) and its particular solution become identical to (41). So, the models are A-stable under $s > 1$. The relation (44) defines additional branch of A-stable models in the region $\alpha > -1/3$ under $s^{**} < s < 1$, where $s^{**} = (1 - \sqrt{3})/2 \approx -0.37$, while relation (47) defines additional branch of A-unstable models under $3/2 > s > 1$.

All results are summarized in diagram of A-stability (fig. 2).

Figure 2: Diagram of A-stability of FRW-models. Inclined dash points region of A-instability of FRW-models. The boundary of this region is double line. At this boundary FRW-models can be A-stable (vertical line $\alpha = 1/3$, horizontal line $s = 1$ and parts of rational dependency (44) with end at the point $\{\alpha^*; s^*\}$ and the beginnings at the points $\{-1/3; s^{**}\}$ and $\{-1; 2\}$ (formula (46)), conditionally A-stable (vertical line (with horizontal dash) $\alpha = -1/3$ under (50); whole performed part of this line corresponds to open models) or A-unstable (part (with inclined dash) of the dependency (47), beginning at the point $\{-1/3; 1\}$; empty point on this branch has coordinates $\{1/3; 5/4\}$ and corresponds to intersection of (47) with vertical line $\alpha = 1/3$, where all models are stable). A-stable branch, which asymptotically tends to line $s = 1/2$, corresponds to $\bar{k} = 0$.

4.2 Structural stability of FRW-models with nonlinear matter

Let consider now the equation (38) with right-hand side of kind $x^{-3(\alpha+1)(b+1)+3}$, responsible for perturbation by nonlinear properties of matter. In the limits $x \to 0$ under $1 + 3\alpha > 0$ this equation can be reduced to the form:

$$2x^2(x\xi(x))'' + (1 - 3\alpha)x(\xi(x) - (x\xi(x))') = x^{-3(1+\alpha)(b+1)+4+3\alpha}, \tag{53}$$

where $\xi(x) = -\delta(x)/\alpha\sigma c^{b+1}x$. Its particular solution:

$$\xi(x)|_{x\to 0} = \frac{x^{-3b(1+\alpha)}}{3b(-3\alpha + 6\alpha b - 1 + 6b)(1 + \alpha)}. \tag{54}$$

Simple analysis of index shows, that *almost all FRW-models with nonlinear matter are S-stable under $b < 0$, and are S-unstable under $b > 0$* (see also interesting analysis in [2, 21])).

Particular solution to the asymptotic equation (53) under $b = 0$ has the form:

$$\xi(x) = \frac{(3\alpha + 1)\ln x - 2}{(1 + 3\alpha)^2}. \tag{55}$$

We see, that such models are S-unstable. Particular solution to the equation (53) under $b = (1 + 3\alpha)/6(1 + \alpha)$, when denominator of (54) vanishes, takes the form:

$$\xi(x)|_{x \to 0} = -\frac{(3\alpha + 1)\ln x + 2}{(1 + 3\alpha)^2}x^{-(1+3\alpha)/2}, \tag{56}$$

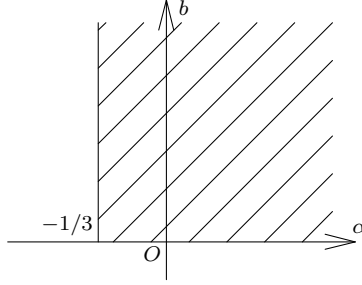and we see, that such models are S-instable too. Results of our analysis are shown in diagram (fig. 3).

Figure 3: Diagram of S-stability of FRW-models with generalized nonlinear equation of state. Inclined dash shows region of S-instability of FRW-models. Boundary belongs to the region.

In case $(1+3\alpha) < 0$ analysis of A-stability is reduced to previous formulas of this section, if one will consider their behaviour under $x \to \infty$. So, from the expression (54) it follows, that *almost all FRW-models with $-1 < \alpha < -1/3$ are A-stable under $b > 0$ and are A-unstable under $b < 0$; almost all FRW-models with $\alpha < -1$ are A-stable under $b < 0$ and are A-unstable under $b > 0$*. Formulas (55)-(56) allow us to conclude, that in all particular cases, which have been considered above, the models are A-unstable.

For the case $1 + 3\alpha > 0$ we obtain the following asymptotic form of equation (38):

$$-2\bar{k}^2x^2(x\xi(x))'' + 6\alpha\bar{k}^2x(\xi(x) - (x\xi(x))') = x^{-3(1+\alpha)(b+1)+3}. \tag{57}$$

Its particular solution has the form:

$$\xi(x)|_{x \to \infty} = -\frac{x^{-1-3\alpha b-3\alpha-3b}}{6b(1 + \alpha)\bar{k}^2(3\alpha b + 3\alpha + 3b + 1)}. \tag{58}$$

Analysis of index shows, that *almost all FRW-models with strong energodominancy condition are A-stable under*

$$b < -(1 + 3\alpha)/3(1 + \alpha) \tag{59}$$

and are A-unstable in case of opposite inequality.

Further analysis of particular cases of the models with strong energodominancy condition reveals that:

- under $b = 0$ models are A-stable;

- under $b = -(1 + 3\alpha)/3(1 + \alpha)$ models are A-unstable;

- under $\bar{k} = 0$ models are A-stable under $b > 0$, while under $b \le 0$ models are A-unstable;

- under $\bar{k} = 0$ and $\alpha = -(6b - 1)/3(2b - 1)$ models are A-stable;

- under $\alpha = -1/3$ models are A-stable and S-unstable when $b > 0$, A-unstable and S-stable when $b < 0$. Under $b = 0$ models are unstable in all senses.

- under $\alpha = -1$ models are A-unstable under any value of b.

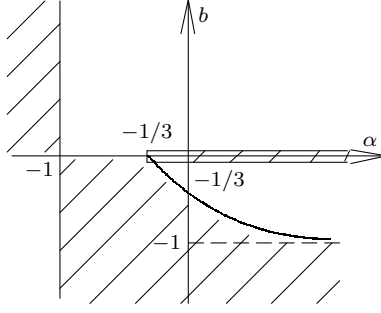All results are summarized in diagram of A-stability (fig. 4).

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Figure 4: Diagram of A-stability of FRW-models with generalized nonlinear equation of state. Inclined dash shows region, where the models are A-unstable. Boundary of this region belongs to the region. Main part of this region on lower half-plane is bounded by the curve (59). Half-line $b = 0$, $\alpha > -1/3$ corresponds to flat A-unstable FRW-models.

5   Stability of standard models with Λ-term

So far we have considered non-perturbed FRW-models with $\Lambda = 0$. One may hope, that extension of class of non-perturbed models will lead to the extension of their stability region. Let us illustrate this idea by example of non-perturbed FRW-models with non-zero Λ-term. For our purposes it will be sufficient to restrict ourselves by FRW-models with dust-like matter ($\alpha = 0$) for generalized nonlinear theories of gravity and by the FRW-models with isotropic radiation ($\alpha = 1/3$) for generalized models with non-linear matter. By note, made after formula (18), inclusion of Λ-term does not change behaviour of FRW-models near singularity, that may be checked directly by our method. The diagram 2 implies, that generalized FRW-models with high orders of scalar curvature under $\alpha = 0$ (and under $\Lambda = 0$) are A-stable under $s \in [1; \infty) \setminus \{7/6\}$ for all values of $\bar{k}^2 \leq 0$ and under $s = (7 - \sqrt{73})/12 \approx -0.13$ for $\bar{k} = 0$. The diagram 4 implies, that generalized FRW-models with isotropic radiation, perturbed by non-linearity of state equation (under $\Lambda = 0$), are A-stable under $b \in [-1/2; \infty)$ for all values of $\bar{k}^2 \leq 0$, excepting the value $b = 0$ under $\bar{k} = 0$.

Standard calculations with using formulas (10)-(17), taken under $\Lambda \neq 0$, lead to the following kind of equation on structural perturbations for models with non-linear gravity under $\alpha = 0$ in the limit $x \to \infty$:

$$x^5 \delta'' + x^4 \delta' - 4x^3 \delta \sim x^4,$$

where we have omitted constant factor in the right-hand side. Its particular solution: $\xi = \delta/x \sim$ const, that means *unconditional A-stability of all FRW-models with Λ-term and dust-like matter.*

For FRW-models with Λ-term and nonlinear equation of state we obtain equation for structural perturbation of the following kind:

$$x^4 \delta'' + 2x^3 \delta' - 6x^2 \delta \sim x^{-1-4b}.$$

Its particular solution: $\xi(x) \sim x^{-4(1+b)}$. So, we conclude, that *due to inclusion of Λ-term, region of A-stability of FRW-models with isotropic radiation become larger: it includes now additional segment of $b \in [-1; -1/2]$ and the value $b = 0$.* This conclusion is closely related to results of papers [16, 17], where generic structural stability of ΛCDM cosmological models has been established by analysis of topological properties of phase portraits of cosmological models.

Complete analysis of structural stability of FRW-models with Λ-term we defer for future publications.

6 Conclusion

We have shown, that the question of structural stability of standard cosmological FRW-models is of non-trivial nature and can be investigated in general manner in frame of outlined classes of generalized cosmological models, including non-linear gravity and non-linear properties of matter. We have restricted ourselves to investigations of structural stability near the cosmological singularity under $a(t) \to 0$ (S-stability) and under $a(t) \to \infty$ (A-stability). Note, that in terms of relative structural perturbation S-stability condition takes the form:

$$\lim_{a \to 0} \frac{\delta}{a} < \infty. \tag{60}$$

So, S-instability condition can be considered in the following two qualitatively different situations. As it can be seen from the content of sections 4 and 5, in almost all cases asymptotic behaviour of structural perturbation is described by power function of a: $\delta \sim a^p$. The condition (60) leads then to the inequality $p \geq 1$, providing S-stability. It is obvious, that under $0 \leq p < 1$ *singularity is conserved, but behaviour of scale factor near singularity changes.* In case $p < 0$, *singularity is destroyed.* In our diagrams of S-stability these situations are indistinguishable. In case of A-instability we, obviously, always deal with change of asymptotic behaviour of scale factor.

Union of all stability diagrams, which corresponds to simultaneous action of structural perturbations of all considered types, leads to the following conclusion: *only the models with $-1 < \alpha < -1/3$, $b > 0$, $s \geq 1$ and the models with $\alpha < -1$, $b < 0$, $s \leq 1$ together with some particular families of cosmological models, having zero measures on diagrams 1-2, possess complete (in frame of chosen class of perturbations) structural stability.* It, in particular, means that *almost all FRW-models with strong energy condition are structurally unstable.*

In our opinion the question of structural stability in cosmology is of principal significance. Consequent account of new observational data and evolution of our theoretical tools of descriptions of nature lead to unavoidable theoretical modifications of basic cosmological equations. If we don't speak about scientific revolution and about change of paradigm, such modifications will be looked at as, in some sense, small corrections in lagrangians or in equations. Our analysis shows that in spite of "smallness", this corrections may lead to the total deleting of one of the properties of original theory, to significant modifications of others properties and to insignificant quantitative variations of the third ones. In the first two cases the properties of original models are "thin" with respect to the modifications of the theory and so, in the context of generalized models, they are "accidental". Our investigation reveals, that cosmological singularity is such "accidental" property of Einstein cosmology. Any non-linear corrections of kind λR^s under $s \geq 2$ in gravitational lagrangian will destroy cosmological singularity, whatever small constant λ is chosen. In fact, cosmological singularity is not experimentally observed. It is desirable *to provide stability of those properties of cosmological models which have robust experimental (observational) support.* In spite of infinity (and even uncountability) of a set of possible generalizations of standard cosmology, the set of its *viable and reasonable* (from the viewpoint of present time) generalizations is quite foreseeable and can be analyzed from general positions [1].

Our example with Λ-term shows that among the set of allowed models one can try to find relatively simple one, which possesses sufficient "structural security" and conserve some important properties under its possible further modifications. Note, that considerations of structural stability can be viewed as purely theoretical argument for inclusion of Λ-term into Einstein theory of gravity. Lets note also that diagrams 2 and 4 throw light on the following question: what class of generalized cosmological models can be relevant to accelerated in latter times universe? The answer is the following: *for FRW models with strong energodominancy condition $(1 + 3\alpha) > 0$ such models must be A-unstable!* In other words relevant generalized models must have in their lagrangians terms of the kind R^s with $s < 1$ and (or) equation of state with nonlinearity index $b < -(1 + 3\alpha)/3(1 + \alpha)$.

Using technic of conform transformations [22] our considerations can be directly applied to the cosmological models within Einstein gravity with self-interacting scalar fields and within non-riemannian geometrical theories. This topic is far from the scope of present paper.

In fact, the question of structural stability is not limited only to cosmology. It is valid in all situations when physics is local and can be described by sets of differential equations. By general considerations any local physical model can be "slightly" modified, such that any forgiven property of the model will be destroyed. Probably, in this situation some fundamental principles will be helpful to restrict the set of possible modifications of the model and to "stabilize" some of its important physical properties.

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